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Dimensional Considerations in Viscoelastic Flows

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I would like to comment from the view of the Noll simple fluid on three topics which have been discussed recently: the definition of characteristic time for a fluid, the significance of the Weissenberg and Deborah numbers, and the solidlike behavior of viscoelastic fluids during rapid accelerations. Before examining these ideas, let me briefly review the concept of the Noll simple fluid. (*Viscoelastic* is used here in the sense that the materials obey neither of the classical linear relations, Newton's law of viscosity and Hooke's law of elasticity. Subclasses of such materials are fluids which show a finite relaxation time and fluids which exhibit normal stresses in viscometric flows (*1*). The term *viscoelastic* is commonly used in the literature in referring to these subclasses.)

THE NOLL SIMPLE FLUID

Let $\boldsymbol{\xi}$ be the place at time $t - s$ $(0 \leq s < \infty)$ of that material particle which at time t occupies the place $\mathbf{x}$

$$\boldsymbol{\xi} = \boldsymbol{\chi}_{(t)}\ (\mathbf{x}, t - s) \tag{1}$$

We call $\boldsymbol{\chi}_{(t)}$ the relative deformation function. By Equation (1) we may describe motion which took place in the material at all times $t - s$ prior to the time t. The gradient with respect to $\mathbf{x}$ of the relative deformation function is called the *relative deformation gradient.*

$$\mathbf{F}_{(t)}\ (t - s) = \nabla\ \boldsymbol{\chi}_{(t)}\ (\mathbf{x}, t - s) \tag{2}$$

The relative right Cauchy-Green strain tensor is defined as

$$\mathbf{C}_{(t)}\ (t - s) = \mathbf{F}_{(t)}\ (t - s)^{\mathrm{T}} \cdot \mathbf{F}_{(t)}\ (t - s) \tag{3}$$

where $\mathbf{F}_{(t)}(t - s)^{\mathrm{T}}$ represents the transpose of $\mathbf{F}_{(t)}(t - s)$.

Noll defines an incompressible simple fluid (*2*) as one for which the stress $\mathbf{T}$ at the position $\mathbf{x}$ and time t is specified within an indeterminate pressure p by the history of the relative right Cauchy-Grees strain tensor for the material which is within an arbitrarily small neighborhood of $\mathbf{x}$ at time t

$$\mathbf{T} + p\,\mathbf{I} = \frac{\mu_o}{s_o} \overset{\infty}{\underset{\sigma=o}{\mathbf{H}^*}}\ (\mathbf{C}_{(t)}\ (t - s_o\sigma)) \tag{4}$$

Here we follow Truesdell's discussion of the dimensional indifference of the definition of a simple material (*2, 3*). The quantity $\overset{\infty}{\underset{\sigma=o}{\mathbf{H}^*}}$ is a dimensionally invariant tensor valued functional. (By tensor valued functional we mean an operator which maps tensor valued functions into tensors.) The constants μ_o and s_o are, respectively, a characteristic viscosity and characteristic time or natural time lapse of the fluid.

DEFINITION OF CHARACTERISTIC TIME OF FLUID

Truesdell (*3*) proposes a definition for the characteristic time s_o which is based upon the three material functions which describe the behavior of incompressible Noll simple fluids in viscometric flows (such as axial flow in a tube or Couette flow). Two of these functions yield normal stress differences for each rate of shear; the third, the shear viscosity function, assigns to each rate of shear what is often referred to as the *apparent viscosity.* His definition is constructed so as to yield a zero characteristic time in the event that the fluid shows no normal stress differences in viscometric flows. Bird (*4, 5*) introduces a characteristic time defined in terms of the shear viscosity function alone. With respect to data correlations, the latter type of definition has a current advantage, since the two material functions which describe normal stress differences are more difficult to measure than the viscosity function. Astarita (*6*) raises the following questions based upon the differences in these definitions.

Does Truesdell's definition imply that the characteristic time must be zero for a fluid which exhibits no normal stress differences, but which does exhibit a variable shear viscosity function? No, inasmuch as Truesdell shows in section 3 and equation (36) of section 6 (*3*), that a fluid with a fading memory, for which the characteristic time is zero, must be a Newtonian fluid. [By fading memory we mean qualitatively that deformations which occurred in the distant past should be less important in determining the present stress than those which occurred in the recent past (*2*). The assumption here is that the real fluids with which we commonly work have a fading memory.] Truesdell's definition for characteristic time is not appropriate for a fluid which does not exhibit normal stress differences, but which does exhibit a variable shear viscosity function in a viscometric flow. [An aqueous 1.05% carboxypolymethylene solution may be such a fluid (*7*), though the Weissenberg effect has been observed in a 2% carboxypolymethylene solution (*8*).] As he pointed out, there are infinitely many possibilities for definitions of characteristic times; there is no reason why every definition should be applicable to every possible situation.

Does not Bird's definition of characteristic time lead to a contradiction in that two fluids might have the same numerical value for the characteristic time, even though they exhibit radically dissimilar normal stress differences in a viscometric flow? No, since the characteristic time and the characteristic viscosity constants do not discriminate among fluids whose stress behavior is described by different functionals $\overset{\infty}{\underset{\sigma=o}{\mathbf{H}}}$ in Equation (4).

Drag reduction in turbulent flow through tubes has been observed for very dilute polymer solutions which exhibit a constant shear viscosity function in viscometric flows (see Figures 4 and 5 of reference 9 and Figure 1 of reference *10*). The fluid does not behave as a Newtonian fluid, and we conclude that the characteristic time cannot be zero. Is Bird's definition for characteristic time inappropriate for this case? Yes. But it is well to keep in mind that Truesdell's definition for characteristic time could fail as well. It is possible that a fluid could behave as a Newtonian fluid in a viscometric flow and yet behave in an entirely different fashion in a nonviscometric flow, such as turbulent flow through a tube. It has been pointed out many times that a full description of the behavior of a fluid in a viscometric flow does not assure us that we would be able to describe the behavior of the material in a nonviscometric flow.

THE WEISSENBERG AND DEBORAH NUMBERS

In their discussions of viscoelastic flows, Reiner (*11*) introduced the Deborah number, N_{De}, and White (*12*) and Metzner, White, and Denn (*13, 14*) proposed the Weissenberg number N_{Wi}:

$$N_{Wi} = s_o V/L \qquad N_{De} = s_o/t_p \tag{5}$$

Here, V, L, and t_p are, respectively, the characteristic velocity, characteristic length, and characteristic time associated with whatever process is under consideration. The implication is that V, L, and t_p are independent parameters for the process. This would be the case for flow in an artery of the human body, where there is a periodic pulse superimposed on the steady flow; but in steady flow through a tube one might take $t_p = L/V$, or in oscillating flow in a manometer tube one could choose $V = L/t_p$. In our remarks below we assume that V, L, and t_p are independent; if they are not, set $N_{De} = N_{Wi}$.

Our purpose here is to show how the Deborah and Weissenberg numbers arise in the context of the Noll simple fluid.

Let us introduce a dimensionless time, velocity, modified pressure, extra stress and gradient operator as follows:

$$t^* = t/t_p,\ \mathbf{v}^* = \mathbf{v}/V,$$
$$\mathcal{P}^* = \mathcal{P}/\mathcal{P}_o,\ (\mathbf{T} + p\,\mathbf{I})^* = \frac{s_o}{\mu_o}(\mathbf{T} + p\,\mathbf{I}), \tag{6}$$
$$\nabla^* = L\,\nabla$$

By modified pressure $\mathcal{P}$ we mean

$$\mathcal{P} = p + \rho\,\Phi \tag{7}$$

where we assume that the external body force vector per unit mass, $\mathbf{f}$, is representable in terms of a potential Φ

$$\mathbf{f} = -\nabla\,\Phi \tag{8}$$

With these definitions and Equation (4), the equation of motion for an incompressible Noll simple fluid may be written as

$$\frac{1}{N_{St}}\frac{\partial \mathbf{v}^*}{\partial t^*} + (\nabla^*\,\mathbf{v}^*)\cdot\mathbf{v}^* = -\frac{1}{N_{Ru}}\nabla^*\,\mathcal{P}^* + \frac{1}{N_{Re}\,N_{Wi}}\operatorname{div}(\mathbf{T} + p\,\mathbf{I})^* \tag{9}$$

Here the Strouhal, Ruark, and Reynolds numbers are defined, respectively, as

$$N_{St} = t_p V/L, \quad N_{Ru} = \rho V^2/\mathcal{P}_o, \quad N_{Re} = \rho VL/\mu_o \tag{10}$$

The product $N_{Re}\,N_{Wi}$ may be thought of as the ratio of the convective inertial terms to the viscous terms in the equation of motion.

Truesdell (*2, 3*) shows that, for an incompressible fluid with fading memory, Equation (4) may be written as

$$(\mathbf{T} + p\,\mathbf{I})^* = \sum_{q=1}^{n} s_o{}^q\,\mathbf{G}_q(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) + 0\,(s_o{}^{n+1}) \tag{11}$$

in which $\mathbf{G}_q$ is the most general isotropic polynomial of degrees $\alpha_1, \ldots, \alpha_n$ in the components of the tensors $\mathbf{A}_1, \ldots, \mathbf{A}_n$ such that

$$\alpha_1 + 2\,\alpha_2 + 3\,\alpha_3 + \ldots + n\,\alpha_n = q \tag{12}$$

The Rivlin-Ericksen tensors are given by (*15*)

$$\mathbf{A}_1{}^* = \frac{L}{V}\mathbf{A}_1 = \nabla^*\,\mathbf{v}^* + (\nabla^*\,\mathbf{v}^*)^T \tag{13}$$

and

$$\mathbf{A}^*{}_{r+1} = \left(\frac{L}{V}\right)^{r+1}\mathbf{A}_{r+1} = \frac{1}{N_{St}}\frac{\partial \mathbf{A}_r{}^*}{\partial t} + (\nabla^*\,\mathbf{A}_r{}^*)\cdot\mathbf{v}^* + (\nabla^*\,\mathbf{v}^*)^T\cdot\mathbf{A}_r{}^* + \mathbf{A}_r{}^*\cdot\nabla^*\,\mathbf{v}^* \tag{14}$$

Equation (11) may be written in a dimensionless form either as

$$(\mathbf{T} + p\,\mathbf{I})^* = \sum_{q=1}^{n}(N_{Wi})\,\mathbf{G}_q{}^*(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n{}^*) + 0\,(N_{Wi}^{n+1}) \tag{15}$$

or as

$$(\mathbf{T} + p\,\mathbf{I})^* = N_{Wi}\left\{\sum_{q=1}^{n}(N_{De})^{q-1}\ {}^{**}(\mathbf{A}_1{}^*, \mathbf{A}_2{}^{**}, \ldots, \mathbf{A}_n{}^{**}) + 0\,(N^n{}_{De})\right\} \tag{16}$$

with

$$\mathbf{A}_{r+1}^{**} = \frac{Lt_p^r}{V}\mathbf{A}_{r+1} = \frac{\partial \mathbf{A}_r^{**}}{\partial t^*} + N_{St}\left[(\nabla^* \mathbf{A}_r^{**})\cdot \mathbf{v}^* + (\nabla^* \mathbf{v}^*)^T \cdot \mathbf{A}_r^{**} + \mathbf{A}_r^{**}\cdot \nabla^* \mathbf{v}^*\right] \quad (17)$$

From Equations (13), (14), and (17), we have that $\mathbf{A}_k^*$ and $\mathbf{A}_k^{**}$ are functions of the Strouhal number, N_{St}, and that they do not depend upon the Deborah and Weissenberg numbers. This, together with Equations (9), (15), and (16), implies that, when the dimensionless constitutive equation for extra stress is inserted in the equation of motion, either the Deborah number or the Weissenberg number appears, but not both. That either the Deborah number or the Weissenberg number is redundant in any given situation (except in the sense that the Strouhal number may appear) is recognized by Metzner, White, and Denn (*14*).

Coleman and Noll (*16*) show that for sufficiently slow processes Equation (4) can be approximated by a sequence of Rivlin-Ericksen fluids. Truesdell (*2, 3*) obtains essentially the same results, Equations (11) and (12), for a fluid with a sufficiently small characteristic time s_o. Equations (15) and (16) serve to consolidate the results of Coleman and Noll with those of Truesdell, since small values of either N_{Wi} or N_{De} may be interpreted as indicating either a slow process or a fluid with a small characteristic time.

AN ILLUSTRATION

One who attempts to use these ideas may feel some hesitation with respect to the definition of characteristic time being left indefinite. This really is no different from the problem we face in defining any characteristic dimension when employing dimensonal analysis, but it may be helpful to consider a specific situation.

Sequences in films prepared by Savins (*17*) and by Markovitz (*18*) show a polymer solution being poured out of a beaker. The beaker is suddenly righted, and the long strand of polymer which had flowed out snaps back into the beaker. The films do not show such a sequence in which the beaker is righted very slowly. But I suggest that, if they had, the polymer would not have returned to the interior of the beaker; it would have drained down the outside from the spout. The time required to right the beaker is a characteristic parameter of this process; the ratio of a characteristic time of the fluid to this characteristic process time is an appropriate Deborah number. When the Deborah number is large (the characteristic time of the fluid is large compared with the characteristic time of the process), memory effects are important in describing the fluid behavior and the fluid appears somewhat solidlike, snapping back into the beaker. When the Deborah number is small, memory effects are not quite so important, and the fluid does not exhibit such pronounced solidlike behavior.

How large must the Deborah number be before the fluid snaps back into the beaker rather than running down its side? This critical Deborah number will be finite and different from zero (we have just seen that a zero Deborah number implies Newtonian behavior); its magnitude will depend upon the definition chosen for the characteristic time of the fluid. Since there are an infinite number of appropriate definitions, we must look at the question in context. There are two possibilities:

1. If one is interested in a single fluid, the characteristic time of the fluid is some constant of no particular interest. Rather than asking how large must the Deborah number be, one asks how small must be t_p.

2. One wishes to correlate this critical Deborah number for a series of fluids which are known to exhibit the same functional $\overset{\infty}{\underset{\sigma=0}{H^*}}$ in Equation (4) and differ only in the magnitudes of their characteristic times and characteristic viscosities. Now one must make a choice for the definition of the characteristic time of the fluid, but the fact that the choice is arbitrary causes no difficulty. Any definition of characteristic time appropriate to the fluid will give a self-consistent correlation of practical interest. The answer as to how large the critical Deborah number is for any particular fluid can be given in this sense.

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NOTATION

$\mathbf{A}_k$ = k^{th} Rivlin-Ericksen tensor, Equations (13) and (14)
$\mathbf{C}_{(t)}(T-s)$ = relative right Cauchy-Green strain tensor, Equation (3)
$\mathbf{F}_{(t)}(t-s)$ = relative deformation gradient, Equation (2)
$\mathbf{1}$ = identity tensor
L = characteristic length
N_{De} = Deborah number
N_{Re} = Reynolds number
N_{Ru} = Ruark number
N_{St} = Strouhal number
N_{Wi} = Weissenberg number
p = pressure
$\mathcal{P}$ = modified pressure, Equation (7)
$\mathcal{P}_o$ = characteristic modified pressure
s_o = characteristic time or natural time lapse of a fluid
$\mathbf{T}$ = stress tensor
t_p = characteristic time of process
V = characteristic velocity
μ_o = characteristic viscosity
$\boldsymbol{\xi}$ = place at time $t - s$ of that material particle which at time t occupies the place $\mathbf{x}$
$\boldsymbol{\chi}_{(t)}$ = relative deformation function, Equation (1)

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